Suggested solutions for HW2

6.2 Q11

One can consider the function given in Section 6.1 Q10

i.e. $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

Indeed, g is differentiable for all $x \in \mathbb{R}$

For $x \neq 0$, $g(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ is a product of functions which are differentiable at x. Therefore, g is differentiable at $x \neq 0$

For $x = 0$, let $\varepsilon > 0$ be given, choose $\delta = \varepsilon > 0$

Then for any $x \in (-\delta, 0) \cup (0, \delta)$, we have

$$\left| \frac{g(x) - g(0)}{x - 0} - 0 \right| = \left| \frac{x^2 \sin\frac{1}{x^2}}{x} \right|$$

$$= \left| x \sin\frac{1}{x^2} \right|$$

$$\leq |x| < \varepsilon$$

Therefore, g is differentiable at $x=0$ with $g'(0) = 0$

Hence, g is differentiable for all $x \in \mathbb{R}$

In particular, g is differentiable on $(0,1)$ and continuous on $[0,1]$

Since $[0,1]$ is compact, it follows immediately that g is uniformly continuous on $[0,1]$

To show g' is unbounded on $(0,1)$, it suffices to show that for any $M > 0$, there exists $x \in (0,1)$ such that $|g'(x)| \geq M$

Choose $x = \frac{1}{\sqrt{2k\pi}}$ where $k \in \mathbb{N}$ is sufficiently large such that

$\sqrt{2k\pi} > \frac{M}{2}$

In this case, we have $\begin{cases} \frac{1}{x} > \frac{M}{2} \\ \cos\frac{1}{x^2} = 1, \ \sin\frac{1}{x^2} = 0 \end{cases}$

$$
\begin{aligned}
\text{Then } |g'(x)| &= \left|2x \sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}\right| \\
&= |0 - 2\sqrt{2k\pi}| \\
&= 2\sqrt{2k\pi} \\
&= \frac{2}{x} > 2 \cdot \frac{M}{2} = M
\end{aligned}
$$

6.2 Q13

Pick any $a, b \in I$ such that $a < b$

Since I is an interval, $[a, b] \subset I$

Note that f is differentiable on I, in particular, f is continuous on $[a, b]$ and differentiable on (a, b)

Then by Mean Value Theorem, there exists a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b-a)$

Because f' is positive on I, $f'(c) > 0$ and $b - a > 0$

It follows that $f(b) > f(a)$, $\forall\ a < b$, $a, b \in I$

6.3 Q1

For $x \in (a,b)$, $x \neq c$, $f(x) = \frac{f(x)}{g(x)} \cdot g(x)$

Since g is continuous at c, $\lim\limits_{x\to c} g(x)$ exists

By assumption, $B = \lim\limits_{x\to c} g(x) = 0$, $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ exists

Then $A = \lim\limits_{x\to c} f(x)$

$= \lim\limits_{x\to c} \left(\frac{f(x)}{g(x)} \cdot g(x) \right)$

$= \left(\lim\limits_{x\to c} \frac{f(x)}{g(x)} \right) \cdot \left(\lim\limits_{x\to c} g(x) \right)$

$= 0$

6.3 Q2

By assumption, $A = \lim\limits_{x\to c} f(x)$ and $B = \lim\limits_{x\to c} g(x)$ exist

i.e. $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ s.t. $|f(x) - A| < \varepsilon_1$ as $0 < |x-c| < \delta_1$ ①

And $\forall \varepsilon_2 > 0$, $\exists \delta_2 > 0$ s.t. $|g(x) - B| < \varepsilon_2$ as $0 < |x-c| < \delta_2$ ②

(i) When $A > 0$ and $B = 0$

Choose $\varepsilon_1 = \frac{A}{2} > 0$, by ① there exists $\delta_1 > 0$ s.t.

$-\frac{A}{2} < f(x) - A < \frac{A}{2}$ whenever $0 < |x-c| < \delta_1$

i.e. $\frac{A}{2} < f(x) < \frac{3}{2}A$

Now given $M > 0$, choose $0 < \varepsilon_2 < \frac{A}{2M}$ i.e. $\frac{A}{2\varepsilon_2} > M$

Similarly by ②, one can find $\delta_2 > 0$ s.t. $0 < g(x) < \varepsilon_2$ as $0 < |x-c| < \delta_2$

Now let $\delta = \min\{\delta_1, \delta_2\}$, if $0<|x-c|<\delta$

then $\frac{f(x)}{g(x)} > \frac{\frac{1}{2}A}{\varepsilon_2} > M$

Hence, $\lim_{x\to c} \frac{f(x)}{g(x)} = +\infty$

(ii) When $A<0$ and $B=0$

Choose $\varepsilon_1 = -\frac{A}{2} > 0$, by ① there exists $\delta_1 > 0$ s.t.

$\frac{A}{2} < f(x) - A < -\frac{A}{2}$ whenever $0<|x-c|<\delta_1$

i.e. $\frac{3A}{2} < f(x) < \frac{1}{2}A$

Now given $M<0$, choose $0<\varepsilon_2<\frac{A}{2M}$ i.e. $\frac{A}{2\varepsilon_2} < M$

Similarly by ②, one can find $\delta_2 > 0$ s.t. $0<g(x)<\varepsilon_2$ as $0<|x-c|<\delta_2$

Now let $\delta = \min\{\delta_1, \delta_2\}$, if $0<|x-c|<\delta$

then $\frac{f(x)}{g(x)} < \frac{\frac{1}{2}A}{\varepsilon_2} < M$

Hence, $\lim_{x\to c} \frac{f(x)}{g(x)} = -\infty$

6.3 Q10(d)

$\lim_{x\to 0_+} \left(\frac{1}{x} - \frac{1}{\arctan x}\right)$ with $I = (0, +\infty)$ has indeterminate form $\infty - \infty$

Then $\lim_{x\to 0_+} \left(\frac{1}{x} - \frac{1}{\arctan x}\right) = \lim_{x\to 0_+} \left(\frac{\arctan x - x}{x \cdot \arctan x}\right)$

(By L'Hospital's Rule) $= \lim_{x\to 0_+} \left(\frac{\frac{1}{1+x^2} - 1}{\frac{x}{1+x^2} + \arctan x}\right)$

$$= \lim_{x \to 0_+} \left(\frac{-x^2}{x + (1+x^2)\arctan x} \right)$$

$$\text{(By L'Hospital's Rule)} = \lim_{x \to 0_+} \left(\frac{-2x}{1 + 2x\arctan x + 1} \right)$$

$$= \lim_{x \to 0_+} \left(\frac{-x}{1 + x\arctan x} \right)$$

$$= 0$$

6.3 Q13

$\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x}$ has the indeterminate form $\frac{\infty}{\infty}$

Therefore, by L'Hospital's Rule,

$$\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x} = \lim_{x \to (\frac{\pi}{2})-} \frac{\sec^2 x}{\sec x \cdot \tan x}$$

$$= \lim_{x \to (\frac{\pi}{2})-} \frac{\sec x}{\tan x}$$

$$= \left(\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x} \right)^{-1}$$

It follows that $\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x} = 1$ or -1

Moreover, when $x \to (\frac{\pi}{2})-$, $\tan x < 0$ and $\sec x < 0$, then $\frac{\tan x}{\sec x} > 0$

It follows that $\lim_{x \to (\frac{\pi}{2})-} \frac{\tan x}{\sec x} = 1$

On the other hand, $\frac{\tan x}{\sec x} = \tan x \cdot \cos x = \sin x$

Then $\lim\limits_{x\to(\frac{\pi}{2})^-} \frac{\tan x}{\sec x} = \lim\limits_{x\to(\frac{\pi}{2})^-} \sin x = 1$